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Infinitesimal calculus in metric spaces

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Abstract

We study the possibility of defining tangent vectors to a metric space at a given point and tangent maps to applications from a metric space into another metric space. Such infinitesimal concepts may help in analysing situations in which no obvious differentiable structure is at hand. Some examples are presented; our interest arises from hyperspaces in particular. Our approach is simple and relies on the selection of appropriate curves. Comparisons with other notions are briefly pointed out.

MSC: 51K05

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1. Introduction

The essence of mathematics consists in dropping the numerous peculiarities of practical situations in order to delineate simple models. The successes of such a method are numerous. They often require some effort and some acceptance of abstractions. It is only after various attempts and some use that these abstract concepts become as natural as the notions of group or normed vector spaces or differentiable manifolds.

Many incentives have led mathematicians to study spaces which have no differentiable structure, but on which some calculus can be performed; see [2–6,18,20,22,33,39–42,51,62]. Among the fields which require such generalizations are the following topics: differential equations [27,32,34], duality [51,55]..., evolution of domains [1,5,6,58,59], geometry [13,15,17,18,35–37,42,44], image reconstruction [30,48,50], mechanics [43,52–54], morphogenesis [5], nonlinear analysis and optimization [2,19,23,26,33], shape optimization [1,5,12,14,25,28,29,38,63], stochastic problems [46,57], viability and invariance [27,34,60]. Several models exist: Cartesian squares, metric measure spaces [2,3,22,39–41]..., mutational spaces [5,6,27–29]... and their variants [20,49]... with various purposes.

The sole metric structure on a set enables one to introduce some analysis concepts, in particular convexity notions. Whereas the analogies with what occurs on normed vector spaces is alluring, in some cases the results are surprising. For instance, in the Heisenberg group endowed with the so-called Carnot–Heisenberg distance, geodetically convex functions are constant [47].

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Our aim here is to introduce a notion of a tangent cone to a metric space and a notion of a tangent map to a map between metric spaces. Such notions enable one to obtain information about the local behaviors of sets and maps. Our constructions are simple and not too restrictive; they rely on an appropriate selection of arcs. They do not require local compactness of the space, a restriction which is not natural for differential calculus. They allow us to state optimality conditions and they open the way to the study of dynamical systems. We make a short comparison with some other concepts, in particular with the notion of a mutational space [6,33], which has been our starting point. Our aim can be seen as an effort to get rid of uniform estimates while concentrating on simpler conditions. As in [6,58,59], our main motivations are the studies of the space $\mathcal{C}$ of bounded closed convex subsets of a normed vector space E and of the space $\mathcal{K}$ of compact subsets of E, both endowed with the Pompeiu–Hausdorff distance. Such spaces can serve as basic models for shape optimization and image reconstruction.

2. Concepts and examples

In the sequel, an *arc* of a metric space (X, d) is a (not necessarily continuous) map from an interval $I := [0, \theta]$ of $\mathbb{R}$ (for some $\theta > 0$) into X;. Without loss of generality, we can extend it to $\mathbb{R}_+$ by taking it as constant on $[\theta, +\infty[$. The whole family of continuous arcs of X is too large in general for calculus purposes (think of the Peano curve). Therefore, we are led to choose a selection of this family in requiring some conditions in order to detect arcs which are regular enough. The condition we impose means that the triangular inequality along the arc is approximately an equality, making it an approximate geodesic.

Definition 1. An arc $c : \mathbb{R}_+ \to X$ of a metric space (X, d) is said to be (initially) rhythmed if the limit of $t^{-1}d(c(t), c(0))$ exists as $t \to 0_+$. It is called a cadence if it is rhythmed and if for any $a \in \mathbb{R}_+$ one has

$$\lim_{(s,t)\to(a,0_+)} \frac{1}{t}d(c(st), c(t)) = |a-1| \lim_{t\to 0_+} \frac{1}{t}d(c(t), c(0)). \tag{1}$$

In this definition, it is enough to suppose $a \in [0, 1]$, as easily seen.

Example. Let X be a subset of a normed vector space and let $c : \mathbb{R}_+ \to X$ be such that the right derivative $c'_+(0) := \lim_{t\to 0_+}(1/t)(c(t) - c(0))$ exists. Then c is a cadence: setting $q(t) := (1/t)(c(t) - c(0))$, we have $(1/t)d(c(t), c(0)) \to \left\|c'_+(0)\right\|$ as $t \to 0_+$ and

$$\frac{1}{t}\|c(st) - c(t)\| = \|sq(st) - q(t)\| \to \left\|ac'_+(0) - c'_+(0)\right\| = |a-1| \left\|c'_+(0)\right\| \quad \text{as } (s,t) \to (a, 0_+).$$

Example. Let (X, d) be a metric space. Suppose $c \mid [0, \theta]$ is a *metric segment*, i.e. that $d(c(r), c(t)) = d(c(r), c(s)) + d(c(s), c(t))$ for $0 \le r \le s \le t \le \theta$. Then, if it is rhythmed, it is a cadence. In order to see that, let us denote by ℓ the limit of $s^{-1}d(c(s), c(0))$ as $s \to 0_+$. Then, for $a \in (0, 1)$, we have, for $s > 0$ close enough to a

$$\frac{1}{t}d(c(st), c(t)) = \frac{1}{t}d(c(0), c(t)) - \frac{s}{st}d(c(0), c(st)) \to \ell - a\ell$$

as $(s, t) \to (a, 0_+)$; for $a = 0$ relation (1) also holds. For $a \ge 1$ we use the relation

$$\frac{1}{t}d(c(st), c(t)) = \frac{s}{st}d(c(0), c(st)) - \frac{1}{t}d(c(0), c(t)) \to a\ell - \ell$$

when $s \ge 1$. Now if c is continuous and parametrized by arc length it is rhythmed: setting $\ell := d(c(0), c(\theta))$, for any $n \in \mathbb{N} \setminus \{0\}$ and any $r \in \{2^{-n}k\theta : k \in \mathbb{N}, k \le 2^n\}$, one has $d(c(r), c(0)) = r\ell$, so that, by density, this relation also holds for $r \in [0, \theta]$. Metric segments are much used in hyperbolic metric spaces; see [61] and its references. □

The definition we have adopted keeps part of the properties of metric segments in order to select not too wild arcs. More precisely, if $c : [0, \theta] \to X$ is a cadence, then c is approximately a metric segment, in the sense that for any $a \in (0, 1)$, there exists some function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{t\to 0} \mu(t) = 0$ such that

$$d(c(0), c(t)) = d(c(0), c(at)) + d(c(at), c(t)) - t\mu(t)$$

for $t \in [0, \theta]$. In fact, setting $k := \lim_{t\to 0_+} d(c(t), c(0))/t$, one has $d(c(at), c(t))/t \to (1-a)k$ as $t \to 0_+$, hence

$$\frac{1}{t}\left(d(c(0), c(at)) + d(c(at), c(t))\right) \to ak + (1-a)k = \lim_{t\to 0_+} \frac{1}{t} d(c(0), c(t)),$$

so that one can take $\mu(t) := (1/t)[d(c(0), c(at)) + d(c(at), c(t)) - d(c(0), c(t))]$.

Example. If c is a geodesic of a Riemannian manifold, then, for some $\theta > 0$ small enough, $c \mid [0, \theta]$ is a metric segment, and hence is a cadence.

Example. Let (X, d) be a metric space which is also a topological manifold satisfying the following condition: for every $\overline{x} \in X$, there exist a normed vector space E, an open neighborhood U of 0 in E, and a homeomorphism $\varphi : U \to V$ from U onto a neighborhood V of $\overline{x}$ such that for any $\varepsilon > 0$, there exist $\rho > 0$ with $B(0, \rho) \subset U$ for which

$$\forall u, u' \in B(0, \rho) \quad (1-\varepsilon)\left\|u - u'\right\| \le d(\varphi(u), \varphi(u')) \le (1+\varepsilon)\left\|u - u'\right\|.$$

Then, one can check that for any $e \in E$, the arc $c : t \mapsto \varphi(te)$ is a cadence issued from $\overline{x}$.

Example. In exotic metric spaces, cadences may be scarce. In particular, if (X, d) is an ultrametric space, i.e. a metric space such that for any $x, x', x'' \in X$ one has $d(x, x'') \le \max(d(x, x'), d(x', x''))$, then every cadence c is almost constant, in the sense that $d(c(t), c(0)) = t\varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to 0$. In fact, if $\alpha := \limsup_{t\to 0_+} \varepsilon(t)$ is positive, and if $a \in (0, 1]$ one has $d(c(st), c(0)) < d(c(t), c(0))$ for $s, t > 0$ small enough, so that, by a well known property of ultrametric spaces, $d(c(st), c(t)) = d(c(t), c(0)) = t\varepsilon(t)$ and condition (1) implies that $\alpha = |1-a|\,\alpha$, a contradiction.

Example. Suppose there exists a family $(h_v)_{v\in V}$ of semi-groups $h_v : \mathbb{R}_+ \times X \to X$ on a metric space (X, d), parametrized by a normed vector space $(V, \|\cdot\|)$ in such a way that for some $\overline{x} \in X$, one has

$$\forall v \in V \quad \frac{1}{t} d(h_v(t, \overline{x}), \overline{x}) \to \|v\| \quad \text{as } t \to 0_+, \tag{2}$$

$$\forall v \in V,\ \forall t \in \mathbb{R}_+,\ \forall x, x' \in X \quad d(h_v(t, x), h_v(t, x')) = d(x, x'). \tag{3}$$

Recall that $h : \mathbb{R}_+ \times X \to X$ is a semi-group if $h(0, \cdot)$ is the identity mapping and if for any $r, s \in \mathbb{R}_+$, one has $h(s, h(r, \cdot)) = h(r+s, \cdot)$. Then, for each $v \in V$, the arc $c : \mathbb{R}_+ \to X$ given by $c(t) := h_v(t, \overline{x})$ is a cadence. Assumption (2) ensures that c is rhythmed. Now, since h_v is a semi-group, and since by (3) $h_v(r, \cdot)$ preserves distances for every $v \in V, r \in \mathbb{R}_+$, we have for $a \in [0, 1], s \in [0, 1),\ t \in \mathbb{R}_+$:

$$\begin{aligned}\frac{1}{t} d(c(t), c(st)) &= \frac{1}{t} d(h_v(st, h_v(t - ts, \overline{x})), h_v(st, \overline{x})) \\ &= \frac{1-s}{t - st} d(h_v(t - ts, \overline{x}), \overline{x}) \to (1-a)\|v\| \quad \text{as } (s, t) \to (a, 0_+).\end{aligned}$$

In [16], some classical group actions satisfying assumptions similar to (2) and (3) (among others) are studied, giving to some homogeneous spaces a structure of mutational space in the sense of [6]. □

Now let us turn to an attempt to define a kind of tangent space. We first observe that if c_1, c_2 are two arcs of X such that $d(c_1(s), c_2(s))/s \to 0$ as $s \to 0_+$, and if c_1 is rhythmed, then c_2 is rhythmed; if c_1 is a cadence, then c_2 is also a cadence: setting $\varepsilon(s) := d(c_1(s), c_2(s))/s$ one has: $(1/t)d(c_2(t), c_2(0)) \to \lim_{t\to 0_+}(1/t)d(c_1(t), c_1(0))$ and

$$|d(c_2(st), c_2(t)) - d(c_1(st), c_1(t))| \le st\varepsilon(st) + t\varepsilon(t),$$

so that condition (1) is satisfied.

Definition 2. A (virtual) velocity, or (virtual) tangent vector, of a metric space (X, d) at $x \in X$ is an equivalence class of cadences $c : \mathbb{R}_+ \to X$ such that $c(0) = x$ for the relation

$$c_1 \simeq c_2 \quad \text{iff } d(c_1(s), c_2(s))/s \to 0 \text{ as } s \to 0_+.$$

A whizz at x of a metric space (X, d) is an equivalence class of rhythmed arcs issued from x.

We denote by $V(X,x)$ or V_xX the set of velocities of (X,d) at $x \in X$, and by $W(X,x)$ the set of whizz of (X,d) at $x \in X$. If $c : \mathbb{R}_+ \to X$ is a cadence such that $c(0) = x$, we denote by v_c (or $c'(0)$ if there is no risk of confusion) its class in the preceding relation.

The sets $V(X,x)$ and $W(X,x)$ can be given a cone structure by setting, for $v \in W(X,x)$ and $\lambda \in \mathbb{R}_+$:

$$\lambda v := (c_\lambda)'(0)$$

where $c : \mathbb{R}_+ \to X$ is a representant of v (i.e. $c'(0) = v$) and $c_\lambda : \mathbb{R}_+ \to X$ is given by $c_\lambda(t) := c(\lambda t)$. It is easy to check that λv does not depend on the choice of c in the class v. We denote by 0 the class of the constant arc with value x which is clearly a cadence. Moreover, one has

$$(\lambda\mu)v = \lambda(\mu v) \quad \forall \lambda, \mu \in \mathbb{R}_+,\ \forall v \in V(X,x).$$

We set

$$\|v\| = \lim_{t\to 0_+} \frac{1}{t} d(c(t), c(0)),$$

where c is a representant of v; (this definition does not depend on the choice of a representant). Moreover, for $v_1, v_2 \in W(X,x)$, we can set

$$d_W(v_1, v_2) = \limsup_{t\to 0_+} \frac{1}{t} d(c_1(t), c_2(t)),$$

where c_i is a representant of v_i for $i = 1, 2$, since the limsup does not depend on the choices of such representants. The proof of the following result is easy.

Lemma 3. *For $u, v, w \in W(X,x)$, $\lambda \in \mathbb{R}_+$, one has:*

$$\|v\| = d_W(v, 0),$$
$$d_W(u, v) = 0 \iff u = v,$$
$$d_W(u, w) \le d_W(u, v) + d_W(v, w),$$
$$\|\lambda v\| = \lambda \|v\|.$$

Now let us turn to the study of maps between metric spaces.

In the following definition, we say that a mapping $f : X \to Y$ between two metric spaces is *stable at* $x \in X$ (or is Stepanoff at x [31]) if there is some $k \in \mathbb{R}_+$ and a neighborhood V of x in X such that $d_Y(f(v), f(x)) \le k d_X(v, x)$ for each $v \in V$. The infimum of such constants is called the *stability rate* of f at x.

Definition 4. A mapping $f : X \to Y$ between two metric spaces is said to be rhythmed at $x \in X$ if it is stable at x, and if for each rhythmed arc $c : \mathbb{R}_+ \to X$ such that $c(0) = x$ the arc $f \circ c$ is rhythmed.

The mapping f is said to be cadenced at $x \in X$ if it is stable at x and if for each cadence $c : \mathbb{R}_+ \to X$ such that $c(0) = x$ the arc $f \circ c$ is a cadence, and if $f \circ c_1 \simeq f \circ c_2$ whenever c_1 and c_2 are two cadences such that $c_1 \simeq c_2$ and $c_1(0) = c_2(0)$. Then the cadence-derivative of f at x is the map $f'_x : V(X,x) \to V(Y, f(x))$ given by $v \mapsto (f \circ c)'(0)$, where c is a representant of v.

In such a case, one has $\left\| f'_x(v) \right\| \le k \|v\|$, where k is a stability rate of f around x. Note that the condition $f \circ c_1 \simeq f \circ c_2$ whenever c_1 and c_2 are two cadences such that $c_1 \simeq c_2$ and $c_1(0) = c_2(0)$ is automatically satisfied when f is locally Lipschitzian around x.

Remark. The proof of Proposition 8 below shows that when X is an open subset of some Euclidean space, an arc $c : \mathbb{R}_+ \to X$ is a cadence if, and only if, the right derivative of c at zero exists. It follows that the preceding definition is compatible with Definition 1: an arc $f : \mathbb{R}_+ \to Y$ of a metric space is a cadence if, and only if, it is cadenced in the sense of the preceding definition.

Similarly, a function $f : X \to \mathbb{R}$ is cadenced at $x \in X$ if, and only if, it is stable at x and such that for each cadence c of X with $c(0) = x$, the function $f \circ c$ is right differentiable at 0, its right derivative being independent of the choice of c in its class. $\square$

The following proposition is an obvious consequence of the definitions.

Proposition 5. *Let $f : X \to Y$ and $g : Y \to Z$ be mappings between metric spaces. Suppose f is rhythmed (resp. cadenced) at $x \in X$ and g is rhythmed (resp. cadenced) at $f(x)$. Then $h := g \circ f$ is rhythmed (resp. cadenced) at x and*

$$h'_x = g'_{f(x)} \circ f'_x.$$

The preceding notions enable us to give an optimality condition.

Proposition 6 (*Fermat's Rule*). *Suppose $f : X \to \mathbb{R}$ is rhythmed (resp. cadenced) at $x \in X$ and attains a local minimum at x. Then, for any whizz $w \in W(X, x)$ (resp. velocity $v \in V(X, x)$), one has $f'_x(w) \geq 0$ (resp. $f'_x(v) \geq 0$).*

Proof. Given $w \in W(X, x)$ and a rhythmed arc c with whizz $w := w_c$, one has $f(c(t)) \geq f(x)$ for $t > 0$ small enough, since $c(t) \to x$ as $t \to 0$. It follows that $(f \circ c)'(0) \geq 0$. Thus, $f'_x(w) = (f \circ c)'(0) \geq 0$. □

When X is a subset of a normed vector space E, the preceding result is better than the classical Fermat's rule [7, Theorem 6.1.9], since $W(X, x)$ (or even $V(X, x)$) may be larger than the incident tangent cone to X at x (see Lemma 7 below).

Example. Let E be an arbitrary normed vector space of infinite dimension. Given a sequence (e_n) of unit vectors of E without any cluster points, let $f : E \to \mathbb{R}$ be given by $f(e_n/n) = -1/n$, $f(x) = 0$ for $x \in X \setminus \{e_n/n : n \geq 1\}$. The arc $c : [0, 1] \to E$ given by $c(0) = 0$, $c(t) = e_{n+1}/(n+1)$ for $t \in (1/(n+1), 1/n]$ is easily seen to be rhythmed and if w is its whizz, one has $f'_x(w) = -1$, so that 0 is not a local minimizer of f. Such a fact cannot be detected by using tangent vectors to E at 0 as in the classical Fermat's rule.

3. Some basic constructions

Let us consider now some familiar constructions for metric spaces and examine their infinitesimal counterparts.

First, if d' is a metric deduced from a metric d on X by $d' := j \circ d$, where $j : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing subadditive map satisfying $j(0) = 0$ and if j has a non null derivative at 0, then an arc c of (X, d) is rythmed (resp. is a cadence) if, and only if, it is rythmed (resp. a cadence) in (X, d'). Thus, taking $j(t) := \min(t, 1)$ or $j(t) = t/(t + 1)$, one can reduce the study to bounded metric spaces. Clearly, the tangent sets to (X, d) and (X, d') at any point coincide.

If X is a subset of a metric space (W, d_W), and if X is endowed with the induced metric d, then it is clear that an arc c of X is rhythmed (resp. a cadence) if and only if it is rhythmed (resp. a cadence) in (W, d_W). Thus, the tangent set to X at any $x \in X$ can be considered as a subset of the tangent set to W at x, and the canonical injection j of X into W is cadenced at each point. If x is an interior point to W, then j'_x is a bijection between the tangent spaces $V(X, x)$ and $V(W, x)$.

If X is a quotient of a metric space (W, d_W), and if the equivalent classes are closed and such that for any u, v in the same class and any class C of W one has $d_W(u, C) = d_W(v, C)$, where $d_W(u, C) := \inf\{d_W(u, w) : w \in C\}$, then X can be endowed with a metric d by setting $d(x, x') := d_W(w, C')$, where $w \in p^{-1}(x)$, $C' := p^{-1}(x')$, $p : W \to X$ being the canonical projection. Then, if c is an arc of (X, d) which is rythmed, one can find an arc b of (W, d_W) which is rythmed and such that $c = p \circ b$. However, if c is a cadence, the existence of a cadence b of (W, d_W) is not guaranteed in general.

Now suppose X is the product of two metric spaces (V, d_V) and (W, d_W), and that its metric is given by $d = \gamma \circ (d_V, d_W)$, where $\gamma : \mathbb{R}^2_+ \to \mathbb{R}$ is a map null at $(0, 0)$ and such that for any $(r, s) \in \mathbb{R}^2_+$ the directional derivative

$$\gamma'((0, 0); (r, s)) := \lim_{(t, r', s') \to (0_+, r, s)} \gamma(tr', ts')/t$$

exists. Such assumptions are satisfied in the classical cases

$$\gamma_1(r, s) := r + s, \qquad \gamma_p(r, s) := (r^p + s^p)^{1/p}, \qquad \gamma_\infty(r, s) := \max(r, s).$$

Then if a and b are rythmed arcs (resp. cadences) in (V, d_V) and (W, d_W) respectively, it is easy to show that the arc $c := (a, b)$ is rythmed (resp. a cadence) in (X, d_X). Thus, the tangent set to (X, d) at $x := (v, w)$ contains the product of the tangent sets to (V, d_V) and (W, d_W) at v and w respectively.

Let us return to the case of embeddings. In the following lemma, assuming that X is a subset of a normed vector space E, we compare $V(X, x)$ with the cone $T^i(X, x)$ of *incident vectors* (or adjacent vectors in the terminology of [7]) to X at x. Recall that $T^i(X, x)$ is the set of vectors $v \in E$ such that $d(x + tv, X)/t \to 0$ as $t \to 0_+$.

Lemma 7. *Suppose X is a subset of some normed vector space $(E, \|\cdot\|)$ and is endowed with the induced metric. Then, for each $x \in X$, there exists an injection j_x of the set $T^i(X, x)$ of incident vectors to X at x into $V(X, x)$. It is even an isometric embedding.*

Proof. For any $v \in T^i(X, x)$, one can find an arc $c : [0, 1] \to X$ such that $c(0) = x$, $(1/t)(c(t) - c(0)) \to v$ as $t \to 0_+$: it suffices to $c(t) \in X \cap B(x + tv, d(t) + t^2)$, where $d(t) := d(x + tv, X)$. By an observation made above, $t \mapsto x + tv$ being a cadence of E, c is a cadence of E, and hence a cadence of X. If $\overline{c} : [0, 1] \to X$ is another arc such that $\overline{c}(0) = x$, $\overline{c}'(0) = v$, we obviously have $c \simeq \overline{c}$, so that we have a well defined map $j_x : T^i(X, x) \to V(X, x)$. In order to prove that j_x is injective, it suffices to show that it is an isometric embedding. Let $v_1, v_2 \in T^i(X, x)$. There exist arcs $c_1, c_2 : [0, 1] \to X$ such that $c_i(0) = x$, $c_i'(0) = v_i$ for $i = 1, 2$. Then

$$\begin{aligned}\|v_1 - v_2\| &= \left\| \lim_{s\to 0_+} s^{-1}(c_1(s) - x) - \lim_{s\to 0_+} s^{-1}(c_2(s) - x) \right\| \\ &= \lim_{s\to 0_+} s^{-1} \|c_1(s) - c_2(s)\| = d_W(j_x(v_1), j_x(v_2)). \quad \square\end{aligned}$$

Proposition 8. *Let X be a subset of some Euclidean space E. Then the embedding of the set $T^i(X, x)$ of incident vectors to X at x into $V(X, x)$ is an isometric embedding onto $V(X, x)$.*

Proof. Let c be a cadence of X such that $c(0) = x$, and let v_c be its class. Let us prove that $(1/t)(c(t) - c(0))$ has a limit as $t \to 0_+$. Then this limit v is an element of $T^i(X, x)$, and its image by j_x in $V(X, x)$ will be v_c by definition of j_x. Let $r := \lim_{t\to 0_+} d(c(t), c(0))/t$. Since the sphere $S(0, r) := \{u \in E : \|u\| = r\}$ is compact, it is enough to show that if v and w are two cluster points of $(1/t)(c(t) - c(0))$ as $t \to 0_+$, then $v = w$. Let (s_n) and (t_n) be two sequences with limit 0 such that $(c(s_n)/s_n) \to v$ and $(c(t_n)/t_n) \to w$. Taking subsequences if necessary, we may suppose $(a_n) := (s_n/t_n)$ has a limit $a \in \mathbb{R}_+ \cup \{\infty\}$. Interchanging the roles of (s_n) and (t_n), we may suppose $a \in [0, 1]$. Then, by (1), we have

$$\begin{aligned}\|av - w\| &= \lim_n \left\| \frac{s_n}{t_n} \frac{c(s_n) - c(0)}{s_n} - \frac{c(t_n) - c(0)}{t_n} \right\| = \lim_n \frac{1}{t_n} \|c(a_n t_n) - c(t_n)\| \\ &= (1 - a) \lim_{t\to 0_+} \frac{1}{t} d(c(t), c(0)) = (1 - a)r.\end{aligned}$$

Thus $\|av - w\|^2 = (1 - 2a + a^2)r^2$, while an expansion using the scalar product gives $\|av - w\|^2 = a^2 \|v\|^2 - 2a(v \mid w) + \|w\|^2 = a^2 r^2 - 2a(v \mid w) + r^2$. It follows that $(v \mid w) = r^2$, and hence $w = v$ since $\|v\| = \|w\| = r$. $\square$

4. Spaces of subsets

The following concrete example is part of our motivation.

Let E be a normed vector space (n.v.s.) and let $\mathcal{X}$ be the set $\mathcal{C}$ of nonempty bounded closed convex subsets of E equipped with the Pompeiu–Hausdorff distance d defined by

$$\begin{aligned}&d(A, B) := \max(e(A, B), e(B, A)) \quad \text{for } A, B \in \mathcal{X}, \text{ where} \\ &e(A, B) := \sup_{a\in A} d(a, B).\end{aligned}$$

For a subset F of E, let h_F be the support function of F given by

$$h_F(u^*) := \sup\{\langle u^*, x\rangle : x \in F\} \quad u^* \in U^*,$$

where U^* is the closed unit ball of the dual E^* of E. Given $A, B \in \mathcal{X}$, let $C : [0,1] \to \mathcal{X}$ be given by

$$C(t) := (1-t)A + tB.$$

Since Hörmander's theorem ([21, Thm II-18]) asserts that for any $A, B \in \mathcal{X}$, one has

$$d(A,B) = \sup_{u^* \in U^*} \left| h_A(u^*) - h_B(u^*) \right|,$$

and since $h_{C(t)} = (1-t)h_A + th_B$, for $s \in (0,1)$ and $s = 0$, we get that

$$\begin{aligned} d(C(st), C(t)) &= \sup_{u^* \in U^*} \left| (1-st)h_A(u^*) + sth_B(u^*) - (1-t)h_A(u^*) - th_B(u^*) \right| \\ &= |1-s|\, t \sup_{u^* \in U^*} \left| h_A(u^*) - h_B(u^*) \right| = |1-s|\, t d(A,B), \\ d(C(0), C(t)) &= td(A,B). \end{aligned}$$

Thus, $d(C(st), C(t)) = |1-s|\, d(C(0), C(t))$, and C is a cadence. In fact, since the Hörmander mapping $h : F \mapsto h_F$ maps $\mathcal{X}$ isometrically into the space $\mathcal{H}$ of positively homogeneous continuous functions on the unit ball of the dual space E^* endowed with the norm of uniform convergence, and since $t \mapsto h_{C(t)}$ is an affine segment, we see that C is a metric segment.

Lemma 7 shows that the velocity $V(\mathcal{X}, A)$ is rich enough, since it contains the image of the set $T^i(h(\mathcal{X}), h(A))$ in $\mathcal{H}$. The latter contains the set $\{h_B - h_A : B \in \mathcal{X}\}$, which are the initial velocities of the curves $t \mapsto h_{(1-t)A+tB}$.

Now, let $\mathcal{X}$ be the set $\mathcal{K}$ of nonempty compact subsets of E endowed with the Hausdorff–Pompeiu metric. Let $BC(E, \mathbb{R})$ be the set of bounded continuous functions on E endowed with the norm of uniform convergence. The map $j : \mathcal{K} \to BC(E, \mathbb{R})$ given by $j(A)(e) = d_A(e) - \|e\|$ being an embedding of $\mathcal{K}$, it may give rise to a set of velocities which is rich enough. Another means to get velocities consists in assuming that A is regular enough to admit deformations $h_t : A \to X$ for $t \in [0,1]$ such that $h_0 = I_A$, the identity mapping of A, and that for each $a \in A$, the derivative $v(a) = \lim_{(t,x)\to(0_+,a)}(1/t)(h_t(x) - x)$ exists, is a continuous function of a, and is normal to A at a. In such a case, for each $a \in A$ one has $d_A(h_t(a)) = t(\|v(a)\| + \varepsilon_a(t))$ with $\varepsilon_a(t) \to 0$ as $t \to 0$. Setting $C(t) := h_t(A)$, we obtain a rhythmed arc of $\mathcal{K}$. In fact,

$$\begin{aligned} \liminf_{t\to 0_+} \frac{1}{t} d(C(t), C(0)) &\geq \liminf_{t\to 0_+} \sup_{a\in A} \frac{1}{t} d(h_t(a), A) \\ &\geq \sup_{a\in A} \sup_{\tau>0} \inf_{t\in(0,\tau)} \frac{1}{t} d(h_t(a), A) \\ &= \sup_{a\in A} \|v(a)\|. \end{aligned}$$

On the other hand, if (t_n), (a_n) are sequences of $(0,1]$ and A respectively such that

$$\limsup_{t\to 0_+} \frac{1}{t} d(C(t), C(0)) = \lim_n \frac{1}{t_n} d(h_{t_n}(a_n), A), \quad \text{or} \quad \lim_n \frac{1}{t_n} d(a_n, h_{t_n}(A))$$

taking a subsequence of (a_n) which converges to some $a \in A$, and relabelling it, we deduce from our assumptions that

$$\limsup_{t\to 0_+} \frac{1}{t} d(C(t), C(0)) = \lim_n \frac{1}{t_n} d(h_{t_n}(a_n), A) \leq \lim_n \frac{1}{t_n} \left\| h_{t_n}(a_n) - a_n \right\| = \|v(a)\|,$$

so that

$$\limsup_{t\to 0_+} \frac{1}{t} d(C(t), C(0)) \leq \sup_{a\in A} \|v(a)\|$$

and $(1/t)d(C(t), C(0))$ has a limit. Another possible means to get rhythmed arcs of $\mathcal{K}$ is to take enlargements, as in [56]. The enlargements of a subset A of X are defined by $A_r := \{x \in X : d_A(x) \leq r\}$ for $r \in \mathbb{R}_+$. The intermediate value theorem ensures that $d(A_r, A) = r$, so that $C : \mathbb{R}_+ \to \mathcal{X}$ for $\mathcal{X} = \mathcal{C}$ or $\mathcal{X} = \mathcal{K}$ given by $C(t) = A_t$.

Moreover, since $d(C(st), C(t)) = t - st$ by [56, Lemma 29 (b)], we see that C is a cadence. On the other hand, the ε-regularization $A_{(\varepsilon)}$ of A as defined in [45] is a curve of $\mathcal{X}$ which may satisfy $d(A_{(\varepsilon)}, A) = 0$ or $d(A_{(\varepsilon)}, A) = +\infty$.

5. Comparisons and open questions

An important stream of results has recently appeared with the study of measure-theoretic definitions of a tangent space. In particular, a tangent space to a locally compact doubling metric space can be defined as a limit in the Gromov–Hausdorff distance of a parametrized family of metric spaces (see [4,37,40, 2.12] and the references therein). Such a tangent space is not unique. A comparison with our set of velocities is not obvious, because here we do not assume local compactness or the existence of a doubling measure. A junction of the two approaches would be of interest.

The main drawback of the preceding notions consists in the impossibility of comparing or relating the sets of velocities to X at different points. When using an isometric embedding $e : X \to E$ of X into a Banach space E, one gets subsets $T^i(e(X), e(x))$ of E which can be related. However, without additional assumptions, it is not clear that one has invariance under different embeddings of X into a Banach space of the incident cones to the images. Thus, one is led to look for canonical embeddings. Besides special embeddings such as the ones we have considered in the section devoted to power sets, the Banach, the Fréchet and the Urysohn embeddings which are limited to separable metric spaces, there is a general class of embeddings, known as the Kuratowski class, which can be used. In order to describe this class, for $y \in X$, let us denote by d_y the function $d(\cdot, y)$, and let us introduce the space $E := BC(X)$ of bounded continuous functions on X, which is a Banach space when endowed with the norm $\|\cdot\|_\infty$ of uniform convergence. Given $y \in X$, let $e_y : X \to E$ be the mapping given by

$$e_y(x)(\cdot) := d_x(\cdot) - d_y(\cdot).$$

It is easy to show that the mapping $e_y : x \mapsto e_y(x)$ is an isometric embedding of (X, d) into $(E, \|\cdot\|_\infty)$. Given two points y, z of X, the mapping e_z is obtained from the mapping e_y by a translation:

$$e_z(x)(w) = e_y(x)(w) + d_y(w) - d_z(w) \quad w, x \in X,$$

or $e_z(x) = e_y(x) + h_{y,z}$ for $x \in X$, where $h_{y,z} := d_y - d_z$ is an element of E independent of x. It follows that $e_z(X) = e_y(X) + h_{y,z}$

$$T^i(e_z(X), e_z(x)) = T^i(e_y(X), e_y(x)) \quad \forall x \in X.$$

In the sequel, we fix $z \in X$ and we consider $T^i(e_x(X), 0) = T^i(e_x(X), e_x(x)) = T^i(e_z(X), e_z(x))$ as a subset of $V(X, x)$ using the isometric embedding of Lemma 7.

Proposition 9. *Let w be a vector field on a complete metric space X, i.e. the data for each $x \in X$ of an element $w(x)$ of $V(X, x)$. Suppose that for each $x \in X$, one has $w(x) \in T^i(e_z(X), e_z(x))$ and that w is locally Lipschitzian from X into E. Then, for any $x_0 \in X$, there exists a cadence c of X satisfying $c(0) = x_0$ and $c'_+(t) = w(c(t))$.*

More precisely, one can assert the existence of an arc $c : [0, \theta) \to X$ such that for each $t \in [0, \theta)$, the arc $c_t : [0, \theta - t) \to X$ given by $c_t(s) := c(s + t)$ is a cadence and $(c_t)'(0) = w(c(t))$. The abuse of notation used in the last equality of the statement is justified by the fact that c can be considered as an arc of $e_z(X)$ which is right differentiable and whose right derivative $c'_+(t)$ at t corresponds to $w(c(t))$ when embedding $T^i(e_z(X), e_z(c(t)))$ into $V(X, c(t))$.

Proof. We may identify X with $e_z(X)$. Then we apply the classical Nagumo–Brezis invariance theorem for vector fields which are tangent to a closed subset. □

Another means to relate the sets of velocities to a metric space (X, d) at different points is to select a class of homotopies of X, i.e. a class of continuous maps $h : X \times [0, 1] \to X$ such that $h(x, 0) = x$ for each $x \in X$. Such an approach is adopted in [5,6,20] and in the works using mutational spaces. Let us note, however, that in such contributions the arcs $t \mapsto h(x, t)$ are not supposed to be cadences (nor even rhythmed); on the other hand, strong uniform estimates are required. It would be interesting to study whether some compromises between the two frameworks would bring new results.

Finally, let us take a step towards one of the earliest devices to get calculus results in metric spaces.

Definition 10. Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be cadenced at $x \in X$. Then the slope of f at x in the direction $v \in V(X, x)$ is $|\nabla_v|(f) := \max(-f'_x(v), 0)$.

The directional slope of f at x is $|\nabla|(f)(x) := \sup\{|\nabla_v|(f) : v \in V(X, x),\ \|v\| = 1\}$.

Recall that the (strong) slope of f at x is defined by

$$\|\nabla\|(f)(x) := \limsup_{x'(\neq x)\to x} \frac{\max(f(x) - f(x'), 0)}{d(x, x')}.$$

Such a notion has been used with much success for dealing with the existence of curves satisfying decrease properties, error bounds estimates and metric regularity (see [8–11,23,24] and their references). The following comparison is easy.

Proposition 11. *Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be cadenced at $x \in X$. Then the directional slope of f at x is majorized by the slope of f at x:*

$$|\nabla|(f)(x) \le \|\nabla\|(f)(x).$$

Proof. Given $v \in V(X, x)$, $\|v\| = 1$ and a representant c of v, we have $d(c(t), c(0))/t \to 1$ as $t \to 0_+$. Therefore

$$\begin{aligned}|\nabla_v|(f) &= \max\left(-\lim_{t\to 0_+} \frac{f(c(t)) - f(c(0))}{t}, 0\right) = \lim_{t\to 0_+} \frac{1}{t}\max(f(x) - f(c(t)), 0)\\ &\le \limsup_{x'(\neq x)\to x} \frac{\max(f(x) - f(x'), 0)}{d(x, x')} = \|\nabla\|(f)(x).\end{aligned}$$

Taking the supremum over $v \in V(X, x)$ satisfying $\|v\| = 1$, we get the required result. □

A consequence of the preceding inequality is that decrease results and metric regularity results can be obtained with the help of the notion of directional slope. In particular, we dispose of the following Decrease Principle and Error Bound Property.

Theorem 12 (*Decrease Principle*). *Let $f : X \to \mathbb{R}_+ \cup \{+\infty\}$ be a nonnegative l.s.c. proper function on a complete metric space X, and let $S := \{x \in X : f(x) = 0\}$. Suppose there are $\overline{x} \in \operatorname{dom} f$, $c > 0$ and $r \in \mathbb{R}_+ \cup \{+\infty\}$ with $f(\overline{x}) < cr$ such that $|\nabla|(f)(u) \ge c$ for any $u \in B(\overline{x}, r) \setminus S$. Then S is nonempty and*

$$d(\overline{x}, S) \le c^{-1} f(\overline{x}).$$

In particular, if for some positive number c, one has $|\nabla|(f)(u) \ge c$ for every $u \in X \setminus S$, then S is nonempty, and for each $x \in X$ one has

$$d(x, S) \le c^{-1} f(x). \tag{4}$$

Proof. Since $|\nabla|(f)(\cdot) \le \|\nabla\|(f)(\cdot)$, the assumption ensures that $\|\nabla\|(f)(u) \ge c$ for any $u \in B(\overline{x}, r) \setminus S$. Then the conclusion follows from the Decrease Principle using the strong slope [8,9]. □

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